PUTNAM PRACTICE SET 10

PROF. DRAGOS GHIOCA

Problem 1. A sequence $\{x_n\}_{n\ge 0} \subset \mathbb{R}$ is called convex if $2x_n \le x_{n-1} + x_{n+1}$ for each $n \ge 1$. Let $\{b_n\}_{n\ge 0} \subset \mathbb{R}_+$ be a sequence with the property that for each positive real number a, we have that the sequence $\{a^n b_n\}_{n\ge 0}$ is convex. Then prove that the sequence $\{\log(b_n)\}_{n\ge 0}$ is also convex.

Solution. Let $n \ge 1$ be given. So, we know that for any $a \in \mathbb{R}_+$, we have that

$$2a^n b_n \le a^{n-1} b_{n-1} + a^{n+1} b_{n+1}.$$

We let $a := \sqrt{\frac{b_{n-1}}{b_{n+1}}}$. Then the above inequality - after dividing by a^n - yields

$$2b_n \le \frac{b_{n-1}}{a} + \frac{b_{n+1}}{a} = 2\sqrt{b_{n-1}b_{n+1}}.$$

So, $b_n^2 \le b_{n-1}b_{n+1}$, for each $n \ge 1$, thus proving the convexity of the sequence $\{\log(b_n)\}_{n\ge 0}$.

Problem 2. Let $P, Q \in \mathbb{R}[x]$ be monic polynomials satisfying the relation: $P^2(x) = (x^2-1)Q^2(x) + 1$ for each $x \in \mathbb{R}$. Prove that $P'(x) = \deg(P) \cdot Q(x)$.

Solution. We first observe by equating the degrees of the two polynomials that it must be that $\deg(P) = 1 + \deg(Q)$ and therefore,

$$\deg(P'(x)) = \deg(Q(x)).$$

Now, differentiating the given identity of polynomials, we obtain

$$2P(x)P'(x) = 2xQ^2(x) + (x^2-1) \cdot 2Q(x)Q'(x)$$

which yields thar $Q(x) \mid P(x) \cdot P'(x)$. However, the relation $P^2(x) = (x^2-1)Q^2(x) + 1$ yields that $\gcd(P(x), Q(x)) = 1$ (i.e., they have no common roots), which means that it must be that $Q(x) \mid P'(x)$. However, since $Q(x)$ and $P'(x)$ have the same degree, we conclude that there exists some constant c such that $P'(x) = c \cdot Q(x)$; finally, equating the leading coefficients in both polynomials $P'(x)$ and $Q'(x)$ yields that c must be $\deg(P)$.

Problem 3. We partition the set $\{1, \dots, 2018\}$ into 6 disjoint subsets $A_1, \dots, A_6$. Prove that there exists some $i \in \{1, \dots, 6\}$ and there exist $x, y, z \in A_i$ (not necessarily distinct) such that $x = y + z$.

Solution. Using the pigeonhole principle, we conclude that there exists some set, say that it is A_1, which contains at least $\frac{2018}{6} > 336$ elements, i.e., there exist $a_1, \dots, a_{337} \in \{1, \dots, 2018\}$ such that $a_i \in A_1$ for each $i = 1, \dots, 337$. Without loss of generality, we assume $a_i < a_j$ for $i < j$. If the desired conclusion does not hold for A_1, then each of the distinct 336 numbers

$$a_{337} - a_1, a_{337} - a_2, \cdots, a_{337} - a_{336} \in \{1, \dots, 2018\}$$

must be contained in one of the sets A_j for $2 \le j \le 6$. So, again without loss of generality, using the pigeonhole principle, there exist at least $\frac{336}{5} > 67$ numbers of the form $a_{337} - a_i$ for some $i \in \{1, \dots, 336\}$ such that $a_{337} - a_i \in A_2$. In other words, we have some

$$336 \ge i_1 > i_2 > \cdots > i_{67} \ge 1$$

such that $b_j := a_{337} - a_{i_j} \in A_2$ for each $j = 1, \dots, 67$. Assume the desired conclusion does not hold for the set A_2; then each of the numbers

$$b_{67} - b_1, b_{67} - b_2, \dots, b_{67} - b_{66} \in \{1, \dots, 2018\}$$

cannot belong to A_2. However, since for each k, we have

$$b_{67} - b_k = a_{i_k} - a_{i_{67}},$$

we conclude that also none of the numbers $b_{67} - b_k$ can be contained in A_1 since otherwise the conclusion would hold for the first set. So, we continue our analysis and then, by the pigenhole principle (without loss of generality), we derive the existence of more than $\frac{66}{4} > 16$ elements of the form $b_{67} - b_k$ contained in A_3. We label these elements (in increasing order) as

$$c_1, \dots, c_{17} \in A_3$$

and note that each of the 16 numbers $c_{17} - c_j$ for $j = 1, \dots, 16$ cannot be contained in A_3, A_2 or A_1, or otherwise the desired conclusion would hold for one of these sets (because each of the numbers $c_{17} - c_j$ are differences of two elements from each one of the sets A_1, A_2 and A_3). So, we continue our process and find that A_4 must contain at least $\frac{16}{3} > 5$ elements of the form $c_{17} - c_k$ for $k \in \{1, \dots, 16\}$. We let

$$d_1 < d_2 < d_3 < d_4 < d_5 < d_6$$

be all these elements and then note that $d_6 - d_i$ for $i = 1, \dots, 5$ is not contained in either one of the first four sets, or otherwise the desired conclusion holds. Hence these 5 elements must be contained in the last two sets which yields (by the pigeonhole principle) that A_5 (say) must contain three elements $e_1 < e_2 < e_3$ (of the form $d_6 - d_i$ for some $1 \le i \le 5$). But then negating the conclusion of our problem we must have that A_6 contains the elements $e_3 - e_1$ and $e_3 - e_2$. But then the element $e_2 - e_1$ cannot be contained in any of the 6 sets, which leads to a contradiction; so, indeed, there must be one of the six sets which contains three elements of the form x, y and $x + y$.

Problem 4. Let $\{F_n\}_{n \ge 0}$ be the Fibonacci sequence, i.e.,

$$F_0 = 0,\ F_1 = 1 \text{ and for each } n \ge 2, \text{ we have } F_n = F_{n-1} + F_{n-2}.$$

Prove that each positive integer m can be written uniquely as a sum of non-consecutive (distinct) elements of the Fibonacci sequence, i.e., there exists $\ell \in \mathbb{N}$ and there exist integers

$$1 < i_1 < \cdots < i_\ell \text{ with } i_{j+1} - i_j \ge 2 \text{ for each } j = 1, \dots, \ell - 1$$

such that $m = F_{i_1} + \cdots + F_{i_\ell}$.

Solution. We prove our result by induction on m; the case $m \le 3$ is obvious as $F_{i+1} = i$ for $i = 1, 2, 3$. So, assume now that the statement holds for all integers $m < M$ (where M is a positive integer larger than three) and next we show that it must also hold for M.

Since $F_n \to \infty$ as $n \to \infty$, then there exists a unique integer $n_0 \geq 4$ (note that $M > F_3$) such that $F_{n_0} \leq M < F_{n_0+1}$. Then let $m := M - F_{n_0}$; if $m = 0$ then we're done, so from now on, we assume that $m \geq 1$. Clearly,

$$m < F_{n_0+1} - F_{n_0} = F_{n_0-1} < F_{n_0} \leq M;$$

so, according to the inductive hypothesis, m can be written (uniquely) as a sum of non-consecutive (distinct) elements of the Fibonacci sequence, i.e., there exists some $\ell \in \mathbb{N}$ and there exist non-consecutive integers

$$1 < i_1 < \cdots < i_\ell < n_0 - 1$$

(note that $m < F_{n_0-1}$) such that

$$m = F_{i_1} + \cdots + F_{i_\ell}.$$

Then $M = F_{i_1} + \cdots + F_{i_\ell} + F_{n_0}$ and also, note that $n_0 - i_\ell \geq 2$, as desired.

Finally, we are left to prove that the above expression of a positive integer as a sum of non-consecutive (distinct) elements of the Fibonacci sequence is actually unique. In other words, we need to prove that if

$$F_{i_1} + \cdots + F_{i_\ell} = F_{j_1} + \cdots + F_{j_k}$$

where $i_{s+1} - i_s \geq 2$ for each $s = 1, \dots, \ell - 1$ and also, $j_{s+1} - j_s \geq 2$ for each $s = 1, \dots, k-1$, then actually we must have that $\ell = k$ and that $i_s = j_s$ for each s. We obtain the desired assertion through induction on the total number of terms in the above equality, i.e., induction on $k+\ell$. Clearly, the result follows when $\ell + k = 2$; so, assume from now on that $k + \ell > 2$. If $j_k = i_\ell$, then we can cancel the last term in both sums and then we are done by the inductive hypothesis. So, assume from now on that $i_\ell \neq j_k$; without loss of generality, we may assume $i_\ell > j_k$. Then we will derive a contradiction because we can show that for any $m \in \mathbb{N}$, we have that

$$F_m > F_{j_0} + F_{j_1} + \cdots + F_{j_k},$$

where $j_0 = 0$, $j_k \leq m-1$ and $j_{i+1} - j_i \geq 2$ for each $i = 0, \dots, k-1$. Indeed, letting $j_{k+1} = j_k + 1$, it suffices to prove that

$$F_{j_{k+1}} > F_{j_0} + F_{j_1} + \cdots + F_{j_k}.$$

For this last inequality, we note that $F_{j_{k+1}} - F_{j_k} = F_{j_k - 1}$ and that

$$j_k - 1 > j_{k-1} > j_{k-1} - 2 \geq j_{k-2} > \cdots > j_2 - 2 \geq j_1 \geq 2$$

and so, the inequality

$$F_m > F_{j_0} + F_{j_1} + \cdots + F_{j_k}$$

follows easily by induction on k. Therefore, we obtain that once we have the equality

$$F_{i_1} + \cdots + F_{i_\ell} = F_{j_1} + \cdots + F_{j_k}$$

(where the sums contain non-consecutive terms in the Fibonacci sequence) then the largest terms in both sums must be equal, i.e, $i_\ell = j_k$. Then the inductive hypothesis on $k + \ell$ delivers the equality term by term in the above two sums, i.e., $k = \ell$ and for each s we must have that $i_s = j_s$. So, indeed, each positive integer m ca be written uniquely as a sum of non-consecutive terms in the Fibonacci sequence:

$$m = F_{i_1} + \cdots + F_{i_\ell} \text{ where}$$
$$2 \leq i_1 \leq i_2 - 2 \leq i_3 - 4 \leq \cdots \leq i_\ell - 2\ell + 2.$$